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Characterization of Ring and Distributive Lattice

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ABSTRACT : In [2] there are some important theorems are related to pseudovariety and Redical class for Group. So in this paper we define pseudovariety and Redical class For Distributive lattice and prove some important results.

Keywords : Characterization of ring, distributive lattice, redical class.

I. INTRODUCTION

Section 1

Before we start the section 2 let's define some definitions which are very helpful in this section.

Definition :

1. Banaschewski measure : A Banaschewski function on a Bounded lattice L is an antitone self-map of L that picks a complement for each element of L.

In an earlier paper *F*. Wehrung proved that every countable complemented modular lattice has a Banaschewski function. And in [19] the same Author prove that there exists a unit-regular ring R of cardinality $\chi 1$ and index of nilpotence 3 such that $L(R)$ has no Banaschewski function, but in this section we prove this for Distributive lattice.

Theorem : If X be a subset in Distributive lattice L with 0. Then a map $\Theta : X^{[2]} \to L$ is a Banaschewski measure then

$$y\Theta x = (y \wedge z)\Theta(y \wedge x), \text{for all } x \le y \le z$$

Proof : Let Θ is a Banaschewski measure on X and also let $x \le y \le z$ in X

And $v := (y \wedge z)\Theta(y \wedge x)$. Obviously $x \wedge v = 0$. Furthermore, as $x \le y$ and By the Definition of Distributive lattice.

$$x \wedge v = y \wedge [(x \vee z)\Theta(x \vee x)] = y \wedge z = y$$

And as $y\Theta x \le v$ and L is Distributive lattice therefore $v = y\Theta x$

Therefore $y\Theta x := (y \wedge z)\Theta(y \wedge x)$

Theorem : Let L is a Distributive lattice with zero, let $e, b \in L$ such that $e \oplus b = 1$. And let [19] $X \subseteq L \downarrow b$. If there exist an L-valued Banaschewski Function on $e \oplus X := \{e \oplus x : x \in X\}$, then there exist a $(L \downarrow b)$-valued [19] Banaschewski Function on X.

Proof : Let Θ is a Banaschewski measure on $e \oplus X$.

Then we have $y\Theta' x := b \wedge [e \vee \{(e \oplus y)\Theta(e \oplus x)\}] = 0$

It is clear that Θ' is $L \downarrow b$-valued and isotone in y and antitone in X.

As L is Distributive lattice therefore

$$x \wedge [e \vee \{(e \oplus y)\Theta(e \oplus x)\}] = 0$$

As $x \le b$ then $x \wedge (y\Theta' x) = 0$

And $x \vee (y\Theta' x) = b \wedge [x \vee e \vee \{(e \oplus y)\Theta(e \oplus x)\}]$

As L is Distributive lattice therefore

$$x \vee (y\Theta' x) = b \wedge (e \vee' y)$$

$$\Rightarrow (b \wedge e) \vee b \wedge y) \Rightarrow y$$

Hence $x \vee (y\Theta' x) = y$

Section 2

In [2] pseudovariety and Redical class are defined for Group. So in this section we define pseudovariety and Redical class for Distributive lattice as follow :

Pseudovariety (in Distributive lattice) : A non-empty class of finite Distributive lattice closed under divisors and finite direct product is called (in Distributive lattice).

Redical class (in Distributive Lattice) : A redical class of finite Distributive lattice is a Subclass with the following properties :

1. It is closed under homo-morphic images.
2. If D is a Distributive Lattice and there are three Normal subgroups (we know that Normal subgroups

form a Distributive lattice [seema] which belong to this class and as the product of these Normal subgroups is also a Normal subgroups. Therefore product of these Normal subgroups also also belongs to this class.

3. For each Lattice this class is unique.

We define pseudovariety and Redical class for Ring as follow :

Pseudovariety (in Ring) : A non-empty class of finite Ring closed under divisors And finite direct product is called (in Ring).

Redical class (in Ring) : A redical class of finite Ring is a Subclass with the following properties :

1. It is closed under homo-morphic images.
2. If R is a Ring and there are Two Normal subgroups which belong to this class , and as the product of these Normal subgroups is also a Normal subgroups. Therefore product of these Normal subgroups also belongs to this class.
3. For each Lattice this class is unique.

Theorem : If R_1 and R_2 are pseudovartites of Ring and let R be a finite Ring, then

1. $R \in R_1.R_2 \Rightarrow \dfrac{R}{R_{11}} \in R_2$

2. $\dfrac{R_{R_1R_2}}{R_{R_1}} = \left(\dfrac{R}{R_1}\right)_{R_2}$

Proof : 1. Let $R \in R_1R_2$ and we have to prove $\dfrac{R}{R_{R_1}} \in R_2$

As R_1 and R_2 are pseudovartites of Ring. Therefore by the definition of pseudovartites $R_1..R_2$ is also a pseudovartites. Let $R_1..R_2 = K$ and as $R \in R_1R_2$ then $R_1..R_2 = K$ must have Normal subgroups. And $K \in R_1$ and $\dfrac{R}{K} \in R_2$.

But we know that by the Definition of Radical $K \subseteq R_{R_1}$ and therefore $\dfrac{R}{R_{R_1}} \in R_{R_2}$.

3. Let I_1 and I_2 are two Ideal of R and $R \in R_1R_2$.

Suppose that $B_i = (I_i)_{R_1}$

As R_1 is pseudovariety then $B_1B_2 \in R_1$ therefore $B_1.B_2 \in R_1.R_2$

Now we will prove $\dfrac{I_1.I_2}{B_1.B_2} \in R_2$

We have $\dfrac{I_1.I_2}{B_1.B_2} = \left(\dfrac{I_1.B_2}{B_1.B_2}\right)\left(\dfrac{I_2.B_1}{B_1.B_2}\right)$

But $\left(\dfrac{I_1.B_2}{B_1.B_2}\right)$ is a ideal of $\dfrac{I_1.I_2}{B_1.B_2}$ and a homo-morphic image of $\dfrac{I_1}{B_1} \in R_2$ and similarly for other factor. The quotient $\dfrac{I_1.I_2}{B_1.B_2} \in R_2$.

Therefore this pseudovarity is a fitting class.

Cor : F_1 and F_2 are pseudovartites of Field [18] and let R be a finite Field, then

1. $F \in F_1.F_2 = \dfrac{F}{F_{F_{11}}} \in F_2$

2. $\dfrac{F_{F_1}}{F_{F_1}} = \left(\dfrac{F}{F_1}\right)_{F_2}$

Proof : As we know that Field has no proper Ideals (it has only F and $\{0\}$) therefore it can be easily proved.

Theorem : If V be an extension-closed pseudovariety of Distributive lattice D containing Ab. If D is a finite Lattice $a \in D_V$ and $b \in D,$ then $\langle a,b\rangle \in V$.

Proof : Suppose that $H = \langle a,b\rangle$ as H is a cyclic extension of Normal subgroups $N = H \cap D_V$ [1]. And as we know that Normal subgroups form a Distributive lattice [seema]. So above will true for Distributive lattice.

As $N \in V, Ab \subseteq V \Rightarrow \langle a,b\rangle \in V$

Theorem : V be an extension-closed pseudovariety of Distributive lattice D containing Ab. And if V-radical admits a binary characterization then

$$D_V = \{a \in D : \forall b \in D, \langle a,b\rangle \in V\}$$

Proof : Let U is a binary characterization of the V-radical and let $a, b \in D.$

Consider the Subgroup $H_b = \langle a,b\rangle$, if $a \in D_V$, then $H_b \in V$ (by above Theorem).

Now, if $H_b \in V : \forall b \in D,$ then $U(a, b) = 1$ for every $u \in U$.

As $U \subseteq (\bar{\Omega}S)^V$ and as U is a characterization, therefore which implies that $D_V = \{a \in D : \forall b \in D, \langle a,b \rangle \in V\}$.

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